

# THE COHOMOLOGY OF $(S(n, k))$ RELEVANT TO MORAVA STABILIZER ALGEBRA

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**ABSTRACT.** In this paper we redefine a increasing filtration on the the Hopf algebra  $S(n, k)$ , From which we get a spectral sequence called May spectral sequence. As an application we computed  $H^{*,*}S(n, n)$  at prime 2,  $H^{*,*}S(3, 2)$  at prime 3 and  $H^{*,*}S(4, 2)$  at prime  $p \geq 5$

## 1. INTRODUCTION

In stable homotopy theory, the “chromatic” point of view plays an important role (*cf.* [3, 11, 14]). Fix a prime  $p$ . Let  $E(n)_*$ ,  $n \geq 0$  be the Johnson-Wilson homology theories and let  $L_n$  be localization functor with respect to  $E(n)_*$ . Then there are natural transformations  $L_n X \rightarrow L_{n-1} X$ , and the chromatic tower

$$\cdots \longrightarrow L_n X \longrightarrow L_{n-1} X \longrightarrow \cdots \longrightarrow L_2 X \longrightarrow L_1 X \longrightarrow L_0 X.$$

By the Hopkins-Ravenel chromatic convergence theorem, the homotopy inverse limit of this tower is the  $p$ -localization of  $X$

$$X \longrightarrow \operatorname{Holim} L_n X.$$

Thus the homotopy groups  $\pi_*(L_n X)$  is the part of homotopy groups  $\pi_*(X)$  one could see from  $E(n)_*$ .

To determine the homotopy groups  $\pi_*(L_n X)$ , one has the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum  $BP$ , whose  $E_2$ -term is

$$E_2^{s,t} = Ext_{BP_* BP}^{s,t}(BP_*, BP_*(L_n X)).$$

(*cf.* [1, 10, 11, 14])

To determine the Adams-Novikov  $E_2$ -term  $Ext_{BP_* BP}^{s,t}(BP_*, BP_*(L_n X))$  one has the Bockstein spectral sequence. This is an argument based on the cohomology of the Morava stabilizer algebra  $S(n)$  at each prime  $p$  (*cf.* [11, 16, 17, 19]). Here the Hopf algebra  $S(n)$  is defined as

$$S(n) = Z/p \otimes_{K(n)_*} K(n)_* K(n) \otimes_{K(n)_*} Z/p,$$

where  $K(n)_* = Z/p[v_n, v_n^{-1}]$ ,

$$K(n)_* K(n) = K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_* = K(n)_*[t_1, t_2, \dots]/(v_n t_s^{p^n} - v_n^{p^s} t_s),$$

$K(n)_*$  acts on  $Z/p$  by sending  $v_n$  to 1. Thus

$$S(n) = Z/p[t_1, t_2, \dots, t_s, \dots]/(t_s^{p^n} - t_s).$$

1991 *Mathematics Subject Classification.* Primary 55Q99; Secondary 55Q51.

*Key words and phrases.* Morava  $n$ -th K-theory, Morava stabilizer algebra, May spectral sequence.

The second author was supported by NSFC grant No. 11471167 and SRFPD No.20120031110025.

We write  $S(n, k) = S(n)/(t_j : j < k) = Z/p[t_k, t_{k+1}, \dots, t_s, \dots]/(t_s^{p^n} - t_s)$ . The Hopf algebra structure of  $S(n)$  determines that of  $S(n, k)$ , while  $S(n, 1) = S(n)$ . Let  $V(n-1)$  and  $T(k-1)$  denote the Smith-Toda spectra and the Ravenel spectra respectively characterized by

$$\begin{aligned} BP_*V(n-1) &= BP_*/I_n = BP_*/(p, v_1, \dots, v_{n-1}) & \text{and} \\ BP_*T(k-1) &= BP_*[t_1, t_2, \dots, t_{k-1}]. \end{aligned}$$

If  $L_nV(n-1) \wedge T(k-1)$  exist, (although  $V(n-1)$  does not exist (*cf.* [6]), but  $V(n-1) \wedge T(k-1)$  might exist), then by the change of rings theorem, the  $E_2$ -term of the Adams-Novikov spectral sequence converging to  $\pi_*(L_nV(n-1) \wedge T(k-1))$  is

$$\begin{aligned} &Ext_{BP_*BP}^{s,t}(BP_*, BP_*(L_nV(n-1) \wedge T(k-1))) \\ &\cong Ext_{S(n,k)}^{s,t}(Z/p, Z/p) \otimes K(n)_*[v_{n+1}, \dots, v_{n+k-1}]. \end{aligned}$$

In this paper, we will use  $H^{s,*}S(n, k)$  to denote the  $Ext$  groups  $Ext_{S(n,k)}^{s,*}(Z/p, Z/p)$ . In [5, 13], Ravenel and Henn determined  $H^{s,*}S(1)$ ,  $H^{s,*}S(2)$  at all primes, and  $H^{s,*}S(3)$  at the odd primes  $p \geq 5$ .  $H^{s,*}S(n, k)$  is known from [11] for  $k \geq n$  at odd primes and  $k > n$  at the prime 2. In [15] Shimomura and Tokashiki computed  $H^{s,*}S(n, n-1)$  at odd primes  $p > 3$ . In this paper we will be concentrated on the case  $k \leq n$ .

Consider the cohomology of the Hopf algebra  $S(n, k)$  at all primes. In section 2 of this paper, we follow Ravenel's ideal (*cf.* [11] 3.2.5 Theorem), redefined the May filtration in  $S(n, k)$  and its cobar complex  $C^{s,*}S(n, k)$ . This filtration induces a spectral sequence so called May spectral sequence  $\{E_r^{s,*M}(n, k), d_r\}$  that converges to  $H^{s,*}S(n, k)$ . Then in section 3 we prove that the  $E_2$ -term of the May spectral sequence is isomorphic to the cohomology of

$$\{E[h_{i,j}|k \leq i \leq s_0, j \in Z/n] \otimes P[b_{i,j}|k \leq i \leq s_0 - n, j \in Z/n], d_1\}$$

where  $s_0 = \max \left\{ \left\lceil \frac{2pn + p - 2}{2(p-1)} \right\rceil, n + k - 1 \right\}$  and  $\left\lceil \frac{2pn + p - 2}{2(p-1)} \right\rceil$  is the integer part of  $\frac{2pn + p - 2}{2(p-1)}$ . In particular, if

$$n + k - 1 \geq \left\lceil \frac{2pn + p - 2}{2(p-1)} \right\rceil,$$

the May's  $E_2$ -term becomes the cohomology of

$$\{E[h_{i,j}|k \leq i \leq n + k - 1, j \in Z/n], d_1\}.$$

The homological dimension of each element is given by

$$s(h_{i,j}) = 1, \quad s(b_{i,j}) = 2.$$

For the May differentials, one has  $d_r : E_r^{s,t,M}S(n, k) \longrightarrow E_r^{s+1,t,M-r}S(n, k)$  and if  $x \in E_r^{s,*}S(n, k)$  then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

The first May differential  $d_1$  is given by

$$d_1(h_{i,j}) = - \sum_{k \leq m \leq i-k} h_{m,j} h_{i-m,j+m}, \quad \text{and} \quad d_1(b_{i,j}) = 0.$$

We analyze the higher May differentials and give a collapse theorem in section 4. As an consequence we compute the cohomology of  $S(n, n)$  at the prime 2,  $S(3, 2)$  at the prime 3 and  $S(4, 2)$  at the prime  $p \geq 5$  in section 5.

## 2. THE MAY SPECTRAL SEQUENCE

Let  $p$  be a prime,  $BP_* = Z_{(p)}[v_1, v_2, \dots]$  and  $BP_*BP = BP_*[t_1, t_2, \dots]$ . For the Hazewinkel's generators described inductively by  $v_s = pm_s - \sum_{i=1}^{s-1} v_{s-i}^{p^i} m_i$  (cf. [4] and [10] 1.2), the diagonal map  $\Delta : BP_*BP \rightarrow BP_*BP \otimes_{BP_*} BP_*BP$  is given by

$$\sum_{i+j=s} m_i(\Delta t_j) = \sum_{i+j+k=s} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}.$$

One can easily prove that

$$\begin{aligned} \Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1 & \text{and} \\ \Delta(t_2) &= \sum_{i+j=2} t_i \otimes t_j^{p^i} - v_1 b_{1,0}, \end{aligned}$$

where  $p \cdot b_{1,0} = \Delta(t_1^p) - t_1^p \otimes 1 - 1 \otimes t_1^p$ . Inductively define

$$p \cdot b_{s,k-1} = \Delta(t_s^{p^k}) - \sum_{i+j=s} t_i^{p^k} \otimes t_j^{p^{i+k}} + \sum_{0 < i < s} v_i^{p^k} b_{s-i,k+i-1},$$

one has

$$(2.1) \quad \Delta(t_{s+1}) = \sum_{i+j=s+1} t_i \otimes t_j^{p^i} - \sum_{0 < i < s+1} v_i b_{s+1-i,i-1}.$$

Thus for the  $n$ -th Morava K-theory  $K(n) = Z/p[v_n, v_n^{-1}]$ , the Hopf algebra

$$K(n)_* K(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$$

is isomorphic to

$$K(n)_* K(n) = Z/p[t_1, t_2, \dots, t_s, \dots] / (v_n t_s^{p^n} - v_n^{p^s} t_s).$$

And  $S(n) = Z/p \otimes_{K(n)_*} K(n)_* K(n) \otimes_{K(n)_*} Z/p$  is isomorphic to

$$S(n) = Z/p[t_1, t_2, \dots, t_n, t_{n+1}, \dots] / (t_s^{p^n} - t_s).$$

The inner degree of  $t_s$  in  $S(n)$  is

$$|t_s| \equiv 2(p-1)(1+p+\dots+p^{s-1}) \pmod{2(p-1)(1+p+\dots+p^{n-1})}$$

because  $v_n$  is sent to 1. The structure map  $\Delta : S(n) \rightarrow S(n) \otimes S(n)$  acts on  $t_s$  as follows

$$\begin{aligned} \Delta(t_s) &= \sum_{0 \leq i \leq s} t_i \otimes t_{s-i}^{p^i} & \text{for } s \leq n \\ \Delta(t_s) &= \sum_{0 \leq i \leq s} t_i \otimes t_{s-i}^{p^i} - b_{s-n,n-1} & \text{for } s > n \end{aligned}$$

where  $b_{i,j} = \sum_{0 < m < p} \binom{p}{m}/p \cdot t_i^{mp^j} \otimes t_i^{(p-m)p^j}$  at odd primes and  $b_{i,j} = t_i^{2^j} \otimes t_i^{2^j}$  at the prime 2. For the integer  $k \geq 1$ , let  $S(n, k) = S(n)/(t_s | s < k)$ . We have

$$S(n, k) = Z/p[t_k, t_{k+1}, \dots, t_{n+k}, t_{n+k+1}, \dots] / (t_s^{p^n} - t_s),$$

the structure map  $\Delta : S(n, k) \longrightarrow S(n, k) \otimes S(n, k)$  acts on  $t_s$  as

$$(2.2) \quad \begin{aligned} \Delta(t_s) &= 1 \otimes t_s + \sum_{k \leq i \leq s-k} t_i \otimes t_{s-i}^{p^i} + t_s \otimes 1 & \text{for } s \leq n+k-1, \\ \Delta(t_s) &= 1 \otimes t_s + \sum_{k \leq i \leq s-k} t_i \otimes t_{s-i}^{p^i} + t_s \otimes 1 - b_{s-n, n-1} & \text{for } s \geq n+k. \end{aligned}$$

In the resulting May spectral sequence, we want to have the 0-th May differential is

$$d_0(t_s^{p^j}) = 0$$

and the first May differential is given by

$$d_1(t_s) = t_k \otimes t_{s-k}^{p^k} + t_{k+1} \otimes t_{s-k-1}^{p^{k+1}} + \cdots + t_{s-k} \otimes t_k^{p^{s-k}} \quad \text{for } s \leq n+k-1,$$

and for  $s \geq n+k$

$$d_1(t_s) = \begin{cases} t_k \otimes t_{s-k}^{p^k} + \cdots + t_{s-k} \otimes t_k^{p^{s-k}} & \text{if the May filtration } M(t_k \otimes t_{s-k}^{p^k}) > M(b_{s-n, n-1}), \\ -b_{s-n, n-1} & \text{if the May filtration } M(t_k \otimes t_{s-k}^{p^k}) \leq M(b_{s-n, n-1}). \end{cases}$$

So we define the May filtration on  $S(n, k)$  as:

**Definition 2.3** In the Hopf algebra  $S(n, k)$ , we define May filtration  $M$  as follows:

- (1) For  $k \leq s \leq n+k-1$ , set the May filtration of  $t_s^{p^j}$  as  $M(t_s^{p^j}) = 2s-1$ .
- (2) For  $n+k \leq s$ , inductively set the May filtration of  $t_s^{p^j}$  as

$$M(t_s^{p^j}) = \max\{2s-1, pM(t_{s-n}^{p^{j+n-1}}) + 1\}.$$

- (3) For the monomial  $t_s^j$ , express  $j$  by the  $p$ -adic number as  $j = j_0 + j_1p + \cdots + j_mp^m$ , where  $0 \leq j_i < p$ . Set the May filtration of  $t_s^j$  as

$$M(t_s^j) = \sum_{0 \leq i \leq m} j_i M(t_s^{p^i}).$$

- (4) And for  $t_{s_1}^{j_1} \cdot t_{s_2}^{j_2} \cdots t_{s_m}^{j_m}$ , where  $s_i \neq s_j$  define its May filtration as

$$M(t_{s_1}^{j_1} \cdot t_{s_2}^{j_2} \cdots t_{s_m}^{j_m}) = \sum_{1 \leq i \leq m} M(t_{s_i}^{j_i}).$$

**Lemma 2.4** Let  $s_0 = \max \left\{ \left\lceil \frac{2pn+p-2}{2(p-1)} \right\rceil, n+k-1 \right\}$  where  $\left\lceil \frac{2pn+p-2}{2(p-1)} \right\rceil$  is the integer part of  $\frac{2pn+p-2}{2(p-1)}$ . Then the May filtration of  $t_s^{p^j}$  satisfies

- (1)  $M(t_s^{p^j}) > M(t_{s-1}^{p^j}) + 1$  and
- (2) For  $s \leq s_0$ , the May filtration  $M(t_s^{p^j}) = 2s-1$ .
- (3) For  $s > s_0$ ,  $pM(t_{s-n}^{p^j}) + 1 \geq 2s-1$  and the May filtration  $M(t_s^{p^j}) = pM(t_{s-n}^{p^j}) + 1$ .

*Proof.* 1) If  $s_0 = \max \left\{ \left\lceil \frac{2pn+p-2}{2(p-1)} \right\rceil, n+k-1 \right\} = n+k-1$ . From its definition, we see that for  $s \leq n+k-1 = s_0$ , the May filtration of  $t_s^{p^j}$  is  $2s-1$  and  $M(t_s^{p^j}) > M(t_{s-1}^{p^j}) + 1$ . From  $n+k-1 \geq \left\lceil \frac{2pn+p-2}{2(p-1)} \right\rceil$ , one sees that

$$s_0 + 1 = n+k > \frac{2pn+p-2}{2(p-1)} \quad \text{and} \quad p(2k-1) + 1 > 2(n+k) - 1.$$

Thus from  $M(t_k^{p^j}) = 2k - 1$ , one knows that the May filtration of  $t_{n+k}^{p^j}$  is  $pM(t_k^{p^j}) + 1$  and

$$M(t_{n+k}^{p^j}) = p(2k - 1) + 1 > 2(n + k) - 1 = M(t_{n+k-1}^{p^j}) + 1.$$

Inductively suppose that  $M(t_s^{p^j}) > M(t_{s-1}^{p^j}) + 1$  and for  $s_0 < s \leq m$ ,

$$pM(t_{s-n}^{p^j}) + 1 > 2s - 1,$$

so the May filtration  $M(t_s^{p^j}) = pM(t_{s-n}^{p^j}) + 1$ . Then from  $M(t_{m+1-n}^{p^j}) > M(t_{m-n}^{p^j}) + 1$  one get

$$\begin{aligned} pM(t_{m+1-n}^{p^j}) + 1 &> p(M(t_{m-n}^{p^j}) + 1) + 1 = pM(t_{m-n}^{p^j}) + p + 1 \\ &> 2m - 1 + p \geq 2(m + 1) - 1. \end{aligned}$$

The May filtration of  $t_{m+1}^{p^j}$  is  $pM(t_{m+1-n}^{p^j}) + 1$ .

If  $s_0 = \left\lfloor \frac{2pn + p - 2}{2(p-1)} \right\rfloor > n + k - 1$ , then for  $k \leq s \leq s_0$ ,  $s \leq \frac{2pn + p - 2}{2(p-1)}$ . This implies

$$p(2(s - n) - 1) + 1 \leq 2s - 1.$$

From  $\frac{2pn + p - 2}{2(p-1)} \leq 2n$  we see that  $s - n \leq n \leq n + k - 1$ . Thus the May filtration

$M(t_{s-n}^{p^j}) = 2(s - n) - 1$  and  $pM(t_{s-n}^{p^j}) + 1 < 2s - 1$ . This implies that the May filtration of  $t_s^{p^j}$  is  $2s - 1$  and  $M(t_s^{p^j}) > M(t_{s-1}^{p^j}) + 1$ .

Notice that  $s_0 + 1 = \left\lfloor \frac{2pn + p - 2}{2(p-1)} \right\rfloor + 1 > \frac{2pn + p - 2}{2(p-1)}$ , this implies

$$p(2(s_0 + 1 - n) - 1) + 1 > 2(s_0 + 1) - 1.$$

The May filtration of  $t_{s_0+1-n}^{p^j}$  is  $2(s_0 + 1 - n) - 1$ , so the May filtration

$$M(t_{s_0+1}^{p^j}) = pM(t_{s_0+1-n}^{p^j}) + 1.$$

Similarly, by induction we get the Lemma.  $\square$

**Example:** The May filtration in  $S(4, 2)$  is given by:

	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$	$t_7$	$t_8$	$t_9$	$t_{10}$	$\dots$
$p = 2$	3	5	7	9	11	13	15	19	23	$\dots$
$p = 3$	3	5	7	9	11	16	22	28	34	$\dots$
$p \geq 5$	3	5	7	9	$3p + 1$	$5p + 1$	$7p + 1$	$9p + 1$	$p(3p + 1) + 1$	$\dots$

Let  $F^{*,M}(n, k)$  be the sub-module of  $S(n, k)$  generated by the elements with May filtration  $\leq M$ . Set  $E^{*,M}(n, k) = F^{*,M}(n, k)/F^{*,M-1}(n, k)$ . One can see from Lemma 2.4 that

$$(2.5) \quad E^{*,*}(n, k) \cong \bigotimes_{k \leq s} T[t_s^{p^j} | j \in Z/n]$$

is a bigraded Hopf algebra, where  $T[\quad]$  denote the truncated polynomial algebra of height  $p$  on the indicated generators. The structure map

$$\Delta : E^{*,*}(n, k) \longrightarrow E^{*,*}(n, k) \otimes E^{*,*}(n, k)$$

acts the the generators  $t_s^{p^j}$  as  $\Delta(t_s^{p^j}) = 1 \otimes t_s^{p^j} + t_s^{p^j} \otimes 1$ .

Let  $C^{s,t}S(n, k) = \otimes^s \bar{S}(n, k)$  denote the cobar construction of  $S(n, k)$  where  $\bar{S}(n, k) = \text{Ker } \epsilon$  denote the augmentation ideal of  $S(n, k)$ . The differential

$$d : C^{s,t}S(n, k) \longrightarrow C^{s+1,t}S(n, k)$$

is given on the generators as

$$(2.6) \quad d(\alpha_1 \otimes \cdots \otimes \alpha_s) = \sum_{1 \leq i \leq s} (-1)^i \alpha_1 \otimes \cdots \otimes (\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i) \otimes \cdots \otimes \alpha_s.$$

In general, the generator  $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_s$  of  $C^{s,t}S(n, k)$  is denoted by  $[\alpha_1 | \alpha_2 | \cdots | \alpha_s]$ . For the generator  $[\alpha_1 | \alpha_2 | \cdots | \alpha_s]$ , define its May filtration as

$$M([\alpha_1 | \alpha_2 | \cdots | \alpha_s]) = M(\alpha_1) + M(\alpha_2) + \cdots + M(\alpha_s).$$

Let  $FC^{*,*,M}S(n, k)$  denote the sub-complex of  $C^{*,*}S(n, k)$  generated by the elements with May filtration  $\leq M$ . Then we get a short exact sequence

$$(2.7) \quad 0 \longrightarrow FC^{*,*,M-1}S(n, k) \longrightarrow FC^{*,*,M}S(n, k) \longrightarrow E_0^{*,*,M}S(n, k) \longrightarrow 0$$

of cochain complexes. The cochain complex

$$E_0^{*,*,M}S(n, k) = FC^{*,*,M}S(n, k) / FC^{*,*,M-1}S(n, k)$$

is isomorphic to the cobar complex of  $E^{*,*}(n, k)$  given in (2.5). Let  $E_1^{*,*,M}S(n, k)$  be the homology of  $(E_0^{*,*,M}S(n, k), d_0)$ . Then (2.7) gives rise to a spectral sequence (so called the May spectral sequence)

$$\{E_r^{s,t,M}(n, k), d_r\}$$

that converges to

$$H^{s,t}(C^{*,*}S(n, k), d) = Ext_{S(n, k)}^{s,t}(Z/p, Z/p).$$

**Theorem 2.8** *For  $k \leq n$  the Hopf algebra  $S(n, k)$  can be given an increasing filtration as in Definition (2.3). The associated bigraded Hopf algebra  $E^{*,*}(n, k)$  is primitively generated with the algebra structure of (2.5). In the associated spectral sequence, the  $E_1$ -term  $E_1^{s,t,M}S(n, k)$  is isomorphic to*

$$E[h_{i,j} | k \leq i, j \in Z/n] \otimes P[b_{i,j} | k \leq i, j \in Z/n].$$

The homological dimension of each element is given by  $s(h_{i,j}) = 1$ ,  $s(b_{i,j}) = 2$  and the degree is given by

$$\begin{aligned} h_{i,j} &\in E_1^{1,2(p^i-1)p^j,*}(n, k), \\ b_{i,j} &\in E_1^{2,2(p^i-1)p^{j+1},*}(n, k) \end{aligned}$$

here  $h_{i,j}$  corresponds to  $t_i^{p^j}$  and  $b_{i,j}$  corresponds to  $\sum \binom{p}{m} / p t_i^{mp^j} \otimes t_i^{(p-m)p^j}$ . One has  $d_r : E_r^{s,t,M}S(n, k) \longrightarrow E_r^{s+1,t,M-r}S(n, k)$  and if  $x \in E_r^{s,t,M}S(n, k)$  then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

In the  $E_1$ -term of this spectral sequence, we have the following relations:

$$h_{i,j} \cdot h_{i_1,j_1} = -h_{i_1,j_1} h_{i,j} \quad h_{i,j} \cdot b_{i_1,j_1} = b_{i_1,j_1} \cdot h_{i,j} \quad b_{i,j} \cdot b_{i_1,j_1} = b_{i_1,j_1} \cdot b_{i,j}$$

*Proof.* It is a routine calculation in homology algebra that for the truncated polynomial algebra  $\Gamma = T[x]$  with  $|x| \equiv 0 \pmod{2}$  and  $x$  primitive,

$$Ext_{\Gamma}(Z/p, Z/p) = E[h] \otimes P[b]$$

where  $h \in Ext^1$  is represented in the cobar complex by  $x$  and  $b \in Ext^2$  is represented by  $\sum \binom{p}{m}/p(x^m \otimes x^{(p-m)})$  ( $b = h^2$  represented by  $x \otimes x$  at the prime 2). Notice that the  $E_0$ -term of the spectral sequence is isomorphic to the cobar complex of  $E^{*,M}(n, k)$ . Then from (2.5) we see that

$$H^{s,*,M}(E_0^{*,t,M}S(n, k), d_0) = Ext_{E^{*,*}(n, k)}^{s,t}(Z/p, Z/p) = \bigotimes_{k \leq s} Ext_{T[t_s^p]}^{*,*}(Z/p, Z/p)$$

Thus the May's  $E_1$ -term

$$E_1^{s,t,M}S(n, k) = E[h_{i,j} | k \leq i, j \in Z/n] \otimes P[b_{i,j} | k \leq i, j \in Z/n].$$

Notice that  $d_0(t_i^{p^j} \cdot t_{i_1}^{p^{j_1}}) = -t_i^{p^j} \otimes t_{i_1}^{p^{j_1}} - t_{i_1}^{p^{j_1}} \otimes t_i^{p^j}$ , we get  $h_{i,j}h_{i_1,j_1} = -h_{i_1,j_1}h_{i,j}$ . In a similar way, one can proof that  $h_{i,j} \cdot b_{i_1,j_1} = b_{i_1,j_1} \cdot h_{i,j}$  and  $b_{i,j} \cdot b_{i_1,j_1} = b_{i_1,j_1} \cdot b_{i,j}$  (cf. [6] Lemma 3.4 and 3.8).  $\square$

### 3. THE FIRST MAY DIFFERENTIALS

Now suppose  $k \leq n$ , then  $s_0 \leq 2n$ . From (2.2) and Lemma 2.4 one has

$$(3.1) \quad \begin{aligned} d_1(h_{i,j}) &= - \sum_{k \leq r \leq i-k} h_{r,j} h_{i-r,j+r} && \text{for } i \leq s_0 \\ d_1(h_{i,j}) &= b_{i-n,j+n-1} && \text{for } s_0 < i. \end{aligned}$$

Thus for  $i > s_0 - n$ ,  $b_{i,j}$  is the boundary of the first May differentials. Recall from [10] Theorem 4.3.22, in the cobar complex of  $BP/I_n$  one has

$$(3.2) \quad d(b_{i,j}) = \sum_{0 < r < i} \left( b_{r,j} \otimes t_{i-r}^{p^{r+j+1}} - t_r^{p^{j+1}} \otimes b_{i-r,r+j} \right)$$

Thus for  $i \leq s_0 - n$ , the first May differential  $d_1(b_{i,j}) = 0$ . This implies:

**Theorem 3.3** *Let  $k \leq n$  and  $s_0$  be given in Lemma 2.4. The  $E_2$ -term of the May spectral sequence is isomorphic to the cohomology of*

$$\tilde{E}_1^{*,*,*}S(n, k) = E[h_{i,j} | k \leq i \leq s_0, j \in Z/n] \otimes P[b_{i,j} | k \leq i \leq s_0 - n, j \in Z/n].$$

The first May differential are given by

$$\begin{aligned} d_1(h_{i,j}) &= - \sum_{k \leq r \leq i-k} h_{r,j} h_{i-r,r+j} && \text{for } i \leq s_0 \\ d_1(b_{i,j}) &= 0 && \text{for } k \leq i \leq s_0 - n \end{aligned}$$

At the prime  $p = 2$ ,  $s_0 = 2n$ . The reduced May  $E_1$ -term becomes

$$\tilde{E}_1^{*,*,*}S(n, k) = E[h_{i,j} | n < i \leq 2n, j \in Z/n] \otimes P[h_{i,j} | k \leq i \leq n, j \in Z/n]$$

and the first May differential of  $h_{2n,j}$  is given by

$$d_1(h_{2n,j}) = - \sum_{k \leq i \leq 2n-k} h_{i,j} h_{2n-i,i+j} + h_{n,j+n-1}^2$$

*Proof.* We define a filtration in the May's  $E_1$ -term

$$E_1^{*,*,*}S(n, k) = E[h_{i,j}|k \leq i, j \in Z/n] \otimes P[b_{i,j}|k \leq i, j \in Z/n]$$

as follows: for each  $s \geq k$ , define

$$F^s(n, k) = \begin{cases} E[h_{i,j}|k \leq i \leq s] & \text{for } k \leq s \leq n+k-1 \\ E[h_{i,j}|k \leq i \leq s] \otimes P[b_{i,j}|k \leq i \leq s-n] & \text{for } n+k-1 < s. \end{cases}$$

From (3.1), we see that for each  $s \geq k$ ,  $F^s(n, k)$  is a sub-complex of  $E_1^{*,*,*}S(n, k)$  that satisfies

$$F^{s_0}(n, k) = \tilde{E}_1^{*,*,*}(n, k) = E[h_{i,j}|k \leq i \leq s_0, j \in Z/n] \otimes P[b_{i,j}|k \leq i \leq s_0-n, j \in Z/n]$$

and

$$F^k(n, k) \hookrightarrow F^{k+1}(n, k) \hookrightarrow \cdots \hookrightarrow F^s(n, k) \hookrightarrow F^{s+1}(n, k) \hookrightarrow \cdots \hookrightarrow E_1^{*,*,*}(n, k).$$

Indeed, for  $s > n+k-1$ ,

$$F^s(n, k) = F^{s-1}(n, k) \bigotimes (E[h_{s,j}|j \in Z/n] \otimes P[b_{s-n,j}|j \in Z/n]).$$

For  $s > s_0$  one has  $d_1(h_{s,j}) = b_{s-n,j+n-1}$ . Thus

$$E[h_{s,j}|j \in Z/n] \otimes P[b_{s-n,j}|j \in Z/n]$$

is a sub-complex of  $F^s(n, k)$  whose cohomology is  $Z/p$  concentrated at dimensional 0. This implies

$$H^*F^{s_0}(n, k) \cong H^*F^{s_0+1}(n, k) \cong \cdots \cong H^*F^s(n, k) \cong \cdots \cong H^*E_1^{*,*,*}S(n, k)$$

At prime  $p = 2$ ,  $s_0 = \left\lfloor \frac{2 \times 2n}{2} \right\rfloor = 2n > n+k-1$ . The first May differentials are deduced from (2.2).  $\square$

As a corollary one can easily see that if  $\frac{2pn+p-2}{2(p-1)} \leq n+k-1$ , then the reduced May's  $E_1$ -term becomes

$$\tilde{E}_1^{*,*,*}S(n, k) = E[h_{i,j}|k \leq i \leq n+k-1, j \in Z/n].$$

**Theorem 3.4** *If  $\frac{2pn+p-2}{2(p-1)} \leq n+k-1$ , then the cohomology of  $S(n, k)$  is of dimensional  $n^2$ .*

#### 4. THE HIGHER MAY DIFFERENTIALS IN THE MSS FOR $S(n, k)$

From (3.2) we see that the first non-trivial May differential of  $b_{i,j}$  appears at

$$(4.1) \quad d_r(b_{i,j}) = \begin{cases} 0 & \text{if } i < 2k. \\ b_{i-k,j}h_{k,j+k+1} - h_{k,j+1}b_{i-k,j+k} & \text{if } i \geq 2k. \end{cases}$$

In [6] (2.10) and (2.11), a collapse theorem is given for the higher May differentials in the exterior part  $E[h_{i,j}|i > 0, j \geq 0]$  of the MSS for the steenrod algebra  $A$  at odd primes. In this section, we will give a similar collapse theorem for the higher May differentials of  $E_r^{*,*,*}S(n, k)$ .

Let  $p$  be an odd prime. We define a Hopf algebra  $T(n, k)$  as

$$(4.2) \quad T(n, k) = P[\xi_i|k \leq i \leq n+k-1].$$



The inner degree of  $\xi_i$  is defined to be  $|\xi_i| = 2(p-1)(1+p+\dots+p^{i-1})$  and the structure map  $\Delta : T(n, k) \longrightarrow T(n, k) \otimes T(n, k)$  acts on  $\xi_i$  by

$$\Delta(\xi_i) = \xi_i \otimes 1 + \sum_{k \leq r \leq i-k} \xi_r \otimes \xi_{i-r}^{p^r} + 1 \otimes \xi_i.$$

There is a Hopf algebra reduction homomorphism  $\Phi : T(n, k) \rightarrow S(n, k)$  which send  $\xi_i$  to  $t_i$ . The image of  $\Phi$  is  $P[t_i | k \leq i \leq n+k-1] / (t_i^{p^n} - t_i)$  and  $\text{Ker } \Phi$  is the idea generated by  $(\xi_i^{p^n} - \xi_i)$ . Further more the homomorphism  $\Phi$  also induces homomorphism in cobar complexes and cohomologies

$$\Phi : \text{Ext}_{T(n, k)}^{s, *} (Z/p, Z/p) \longrightarrow \text{Ext}_{S(n, k)}^{s, *} (Z/p, Z/p)$$

Similar to that of definition 2.3, we set May filtration on  $T(n, k)$  as

$$M(\xi_i^{p^j}) = 2i - 1$$

and let  $F^{*, M}T(n, k)$  be the sub-module of  $T(n, k)$  generated by the elements with May filtration  $\leq M$ . Then  $E^{*, M}T(n, k) = F^{*, M}T(n, k) / F^{*, M-1}T(n, k)$  becomes a bigraded Hopf algebra with the structure of

$$E^{*, *}T(n, k) = \bigotimes T[\xi_i^{p^j} | k \leq i \leq n+k-1, j \geq 0]$$

and  $\Delta(\xi_i^{p^j}) = \xi_i^{p^j} \otimes 1 + 1 \otimes \xi_i^{p^j}$ .

Consider the cobar construction  $C^{s, *}T(n, k)$  of  $T(n, k)$ . Similarly for the generator  $[\beta_1 | \beta_2 | \dots | \beta_s]$  of  $C^{s, *}T(n, k)$  define its May filtration as

$$M([\beta_1 | \beta_2 | \dots | \beta_s]) = M(\beta_1) + M(\beta_2) + \dots + M(\beta_s)$$

and let  $FC^{*, *, M}T(n, k)$  denote the sun-complex generated by elements with May filtration  $\leq M$ . We get a spectral sequence  $\{E_r^{s, *, M}T(n, k), d_r\}$  with  $E_0$ -term

$$E_0^{*, *, M}T(n, k) = FC^{*, *, M}T(n, k) / FC^{*, *, M-1}T(n, k)$$

which is isomorphic to the cobar complex of  $E^{*, *}T(n, k)$ . The  $E_1$ -term of this spectral sequence is isomorphic to

$$(4.3) \quad E_1^{*, *, *}T(n, k) = E[h'_{i,j} | k \leq i \leq n+k-1, j \geq 0] \otimes P[b'_{i,j} | k \leq i \leq n+k-1, j \geq 0].$$

Noticed that the reduction map  $\Phi : T(n, k) \rightarrow S(n, k)$  is May filtration preserving, it induces a homomorphism of May spectral sequences

$$\Phi : E_r^{*, *, *}T(n, k) \longrightarrow E_r^{*, *, *}S(n, k).$$

**Theorem 4.4** *The reduction map  $\Phi : T(n, k) \longrightarrow S(n, k)$  induces a homomorphism between May spectral sequences  $\Phi : E_1^{*, *, *}T(n, k) \longrightarrow E_1^{*, *, *}S(n, k)$  which sends  $h'_{i,j}$  and  $b'_{i,j}$  to  $h_{i,j}$  and  $b_{i,j}$  respectively. It sends infinite cocycles of  $E_r^{*, *, *}T(n, k)$  to that of  $E_r^{*, *, *}S(n, k)$ .*

Similar to [6] (2.10) and (2.11) we give a collapse theorem in the MSS for  $T(n, k)$ . To the generators  $h'_{i,j}, b'_{i,j} \in E_1^{*, *, *}T(n, k)$  define their index as

$$SI(h'_{i,j}) = SI(b'_{i,j}) = i.$$

Given a monomial  $g = x_1 x_2 \dots x_m \in E_1^{*, *, *}T(n, k)$  where each  $x_i$  is of the generators  $h'_{i,j}$  or  $b'_{i,j}$ , define its sum of index as

$$(4.5) \quad SI(g) = SI(x_1) + SI(x_2) + \dots + SI(x_m).$$

For example the sum of index of  $h'_{4,0} h'_{3,0} b'_{2,1}$  is 9

We use  $s(x)$  to denote the homological dimension of  $x$ . Noticed that the May filtration of  $h'_{i,j}$ ,  $b'_{i,j}$  satisfies

$$\begin{aligned} M(h'_{i,j}) &= 2i - 1 = 2SI(h'_{i,j}) - 1 = 2SI(h'_{i,j}) - s(h'_{i,j}) \\ M(b'_{i,j}) &= p(2i - 1) > 2SI(b'_{i,j}) - 2 = 2SI(b'_{i,j}) - s(b'_{i,j}) \end{aligned}$$

we see that for the monomial  $g = x_1 x_2 \cdots x_m \in E_1^{s,*,*}T(n, k)$  of homological dimension  $s$ , its May filtration satisfies

$$\begin{aligned} (4.6) \quad M(g) &= M(x_1) + M(x_2) + \cdots + M(x_m) \\ &\geq 2SI(x_1) - s(x_1) + 2SI(x_2) - s(x_2) + \cdots + 2SI(x_m) - s(x_m) \\ &= 2SI(g) - s \end{aligned}$$

and the equality holds if and only if  $g$  is a monomial in  $E[h'_{i,j} | k \leq i \leq n + k - 1, j \geq 0]$ .

Given an integer  $t = 2(p - 1)(c_0 + c_1 p + \cdots + c_m p^m)$  with  $0 \leq c_i < p$ , we define its sum of degree as

$$Sd(t) = c_0 + c_1 + \cdots + c_m$$

and for an element  $g \in E_1^{s,*,*}T(n, k)$ , express its inner degree  $|g|$  as  $|g| = 2(p - 1)(c_0 + c_1 p + \cdots + c_m p^m)$ , where  $0 \leq c_i < p$  and define its sum of degree to be

$$(4.7) \quad Sd(g) = Sd(|g|) = c_0 + c_1 + \cdots + c_m.$$

Then from

$$\begin{aligned} |h'_{i,j}| &= 2(p - 1)(p^j + p^{j+1} + \cdots + p^{i+j-1}) \\ |b'_{i,j}| &= 2(p - 1)(p^{j+1} + p^{j+2} + \cdots + p^{i+j}) \end{aligned}$$

we see that  $SI(h'_{i,j}) = Sd(h'_{i,j})$ ,  $SI(b'_{i,j}) = Sd(b'_{i,j})$ . But for the reason of the  $p$ -adic numbers one has

$$(4.8) \quad SI(x_1 x_2 \cdots x_s) \geq Sd(x_1 x_2 \cdots x_m).$$

**Theorem 4.9** *In the May spectral sequence for  $T(n, k)$ ,*

(1) *If the inner degree  $t = 2(p - 1)(c_0 + c_1 p + \cdots + c_m p^m)$  and the May filtration*

$$M < 2Sd(t) - s = 2(c_0 + c_1 + \cdots + c_m) - s,$$

*then the May's  $E_1$ -term  $E_1^{s,t,M}T(n, k) = 0$ .*

(2) *If a cocycle  $g \in E[h'_{i,j} | k \leq i \leq n + k - 1, j \geq 0]$  in the exterior part of May's  $E_1$ -term satisfies  $SI(g) = Sd(g)$ , then it is an infinite cocycle in the MSS for  $T(n, k)$  and  $\Phi(g)$  is an infinite cocycle in the MSS for  $S(n, k)$ .*

*Proof.* (1) follows from (4.6) and (4.8).

Suppose  $g \in E[h'_{i,j} | k \leq i \leq n + k - 1, j \geq 0]$  is a cocycle in the exterior part of May's  $E_1$ -term  $E_1^{s,t,M}T(n, k)$  that satisfies  $SI(g) = Sd(g)$ . Then its May filtration  $M = 2SI(g) - s = 2Sd(t) - s$ . Consider the higher May differentials

$$d_r : E_r^{s,t,M}T(n, k) \rightarrow E_r^{s+1,t,M-r}T(n, k),$$

we see that  $M - r < 2Sd(t) - (s + 1)$  for  $r > 1$ . Thus the target  $E_1^{s+1,t,M-r}T(n, k)$  and then  $E_r^{s+1,t,M-r}T(n, k)$  is zero.  $\square$

**Example** Let  $p \geq 5$ . The  $E_2$ -term of the May spectral sequence for  $H^{*,*}S(4, 2)$  is isomorphic to the homology of

$$E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j} | j \in \mathbb{Z}/4]$$

with first May differentials

$$\begin{aligned} d_1(h_{2,j}) &= 0, & d_1(h_{3,j}) &= 0 \\ d_1(h_{4,j}) &= h_{2,j}h_{2,j+2}, & d_1(h_{5,j}) &= h_{2,j}h_{3,j+2} + h_{3,j}h_{2,j+3}. \end{aligned}$$

So  $h_{5,0}h_{4,0}h_{3,0}h_{2,0}$  is a cohomology class in May's  $E_2$ -term.

To prove that  $h_{5,0}h_{4,0}h_{3,0}h_{2,0}$  is a infinite cocycle in the MSS  $E_r^{4,*}S(4, 2)$ , consider the MSS for  $T(4, 2)$ .  $h_{5,0}h_{4,0}h_{3,0}h_{2,0}$  is a 4-dimensional cocycle in the exterior part of May's  $E_1$ -term  $E_1^{4,t,M}T(4, 2)$ .

$$\deg(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}) = 2(p-1)(4+4p+3p^2+2p^3+p^4),$$

$$SI(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}) = 14 = Sd(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}).$$

Thus it is an infinite cocycle in the MSS for  $T(4, 2)$  and  $h_{5,0}h_{4,0}h_{3,0}h_{2,0} = \Phi(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0})$  is an infinite cocycle in the MSS for  $S(4, 2)$ .

## 5. THE COHOMOLOGY OF $S(n, n)$ AT $p = 2$ AND OF $S(3, 2)$ AT $p = 3$

As an application of Theorem 3.3, we will compute  $H^{*,*}S(n, n)$  at  $p = 2$ ,  $H^{*,*}S(3, 2)$  at prime  $p = 3$  and  $H^{*,*}S(4, 2)$  at prime  $p \geq 5$  in this section.

**5.1. The cohomology of  $S(n, n)$  at prime two.** Consider the cohomology of  $S(n, n)$  at  $p = 2$ . The reduced Mays  $E_1$ -term becomes

$$\tilde{E}_1^{*,*,*}S(n, n) = P[h_{n,j} | j \in \mathbb{Z}/n] \otimes E[h_{s,j} | n < s \leq 2n, j \in \mathbb{Z}/n]$$

(cf. Theorem 3.3). Noticed that the only non-trivial first May differential is

$$(5.1) \quad d_1(h_{2n,j}) = h_{n,j-1}^2 + h_{n,j}^2,$$

We see that the  $E_2$ -term is the tensor product of  $E[h_{s,j} | n < s < 2n]$  and the cohomology of

$$\{P[h_{n,j} | j \in \mathbb{Z}/n] \otimes E[h_{2n,j} | j \in \mathbb{Z}/n], d_1\}.$$

**Lemma 5.2** *The May's  $E_2$ -term  $E_2^{*,*,*}S(n, n)$  at  $p = 2$  is isomorphic to the tensor product of  $E[h_{s,j} | n < s < 2n, j \in \mathbb{Z}/n]$  and  $E[h_{n,j}, \rho_{2n} | j \in \mathbb{Z}/n] \otimes P[h_{n,n-1}]$ , where  $\rho_{2n} = \sum_{0 \leq j < n} h_{2n,j}^2$  and  $h_{n,j}^2 = h_{n,n-1}^2$ .*

*Proof.* We define  $b_{n,j} = h_{n,j}^2 + h_{n,j+1}^2$  for  $0 \leq j \leq n-2$  and define  $b_{n,n-1} = h_{n,n-1}^2$ . It is easy to see that  $P[h_{n,j} | j \in \mathbb{Z}/n]$  could be divided as the tensor product of  $P[b_{n,j} | 0 \leq j < n]$  and  $E[h_{n,j} | j \in \mathbb{Z}/n]$  as  $\mathbb{Z}/2$ -modules.

$$P[h_{n,j} | 0 \leq j \leq n-1] = P[b_{n,j} | 0 \leq j \leq n-1] \otimes E[h_{n,j} | 0 \leq j \leq n-1].$$

From (5.1) we see that

$$d_1(h_{2n,j}) = \begin{cases} b_{n,j-1} & \text{if } 1 \leq j < n \\ \sum_{0 \leq i < n-2} b_{n,i} & \text{if } j = n. \end{cases}$$

The cohomology of  $\{P[h_{n,j} | j \in \mathbb{Z}/n] \otimes E[h_{2n,j} | j \in \mathbb{Z}/n], d_1\}$  is isomorphic to the tensor product of  $E[h_{n,j} | j \in \mathbb{Z}/n]$  and the cohomology of

$$P[b_{n,j} | 0 \leq j < n] \otimes E[h_{2n,j} | j \in \mathbb{Z}/n].$$

The generator of  $P[b_{n,j}|j \in Z/n]$  are denoted as

$$b_{n,i_1}^{s_1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m}$$

subject to  $s_i > 0$ ,  $0 \leq i_1 < i_2 < \cdots < i_m < n$  and the generators of  $E[h_{2n,j}|j \in Z/n]$  are denoted as

$$h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1}$$

subject to  $n \geq j_k > \cdots > j_2 > j_1 > 0$ .

For the generators of  $E[h_{2n,j}|0 < j \leq n] \otimes P[b_{n,j}|0 \leq j < n]$  described as above, one has

(1) For  $j_1 > i_1 + 1$ ,

$$h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1} b_{n,i_1}^{s_1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m}$$

is the leading term of the first May differential

$$d_1(h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1} h_{2n,i_1-1} b_{n,i_1}^{s_1-1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m}).$$

While for  $i_1 < n - 1$ ,

$$b_{n,i_1}^{s_1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m}$$

is the leading term of the first May differential

$$d_1(h_{2n,i_1+1} b_{n,i_1}^{s_1-1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m}).$$

(2) For  $j_1 \leq i_1 + 1$  and  $j_1 < n$ , the leading term of the first May differential

$$d_1(h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1} b_{n,i_1}^{s_1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m})$$

is

$$h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1-1} b_{n,i_1}^{s_1} b_{n,i_2}^{s_2} \cdots b_{n,i_m}^{s_m}.$$

(3) For  $j_1 = n = i_1 + 1$ ,

$$d_1(h_{2n,n} b_{n,n-1}^{s_1} \cdot x) = \sum_{i=0}^{n-2} b_{n,i} b_{n,n-1}^{s_1} \cdot x = d_1\left(\sum_{i=0}^{n-2} h_{2n,i+1} b_{n,n-1}^{s_1} \cdot x\right)$$

Thus the cohomology of  $E[h_{2n,j}|j \in Z/n] \otimes P[b_{n,j}|j \in Z/n]$  is isomorphic to  $E[\rho_{2n}|j \in Z/n] \otimes P[b_{n,n-1}]$ , where  $\rho_{2n} = \sum_{0 \leq i < n} h_{2n,i}$ . The Lemma follows.  $\square$

**Theorem 5.3** *The May  $E_\infty$ -term  $E_\infty^{*,*}S(n, n)$  is isomorphic to its  $E_2$ -term. Thus the cohomology of  $S(n, n)$  at prime 2 isomorphic to the tensor product of  $E[h_{s,j}|n < s < 2n]$  and  $E[h_{n,j}, \rho_{2n}|j \in Z/p] \otimes P[h_{n,0}]$*

*Proof.* It is easy to see from (2.2) that for  $n \leq s < 2n$ ,  $h_{s,j}$  is an infinite cocycle. From  $d(t_{2n} + t_{2n}^2 + \cdots + t_{2n}^{2^{n-1}}) = 0$  we get the infinite cocycle  $\rho_{2n}$ . The Theorem follows.  $\square$

**5.2. The cohomology of  $S(3, 2)$  at prime 3.** Now consider the cohomology of  $S(3, 2)$  at prime  $p = 3$ . From Lemma 2.4 we see that the  $s_0 = 4$ . Thus from Theorem 3.3 we see that the reduced May's  $E_1$ -term is

$$\tilde{E}_1^{*,*}S(3, 2) = E[h_{2,j}, h_{3,j}, h_{4,j}|j \in Z/3]$$

and the first May differentials are given by

$$(5.4) \quad \begin{aligned} d_1(h_{2,j}) &= 0 & d_1(h_{3,j}) &= 0 & \text{and} \\ d_1(h_{4,j}) &= -h_{2,j}h_{2,j+2}. \end{aligned}$$

The May's  $E_2$ -term is isomorphic to

$$E_2^{*,*,*}S(3, 2) = H^{*,*,*}(E[h_{2,j}, h_{4,j}|j \in Z/3], d_1) \otimes E[h_{3,j}|j \in Z/3]$$

**Lemma 5.5** *The May  $E_2$ -term for  $H^*S(3, 2)$  at  $p = 3$  has Poincare series  $(x^6 + 3x^5 + 6x^4 + 9x^3 + 6x^2 + 3x + 1)(x + 1)^3$ . It is the tensor product of  $E[h_{3,j}|j \in Z/3]$  and the  $Z/3$  module  $\mathcal{C}$  generated by the following element.*

The generators of $\mathcal{C}$							
Dimension	0	1	2	3	4	5	6
Generators	1	$h_{2,j}$	$g_j$ $k_j$	$l_j$ $l'_j$ $k_j h_{2,j}$	$g_j g_{j+1}$ $k_j k_{j+1}$	$g_j l_{j+1}$	$A$

where  $j \in Z/3$ ,  $g_j = h_{4,j}h_{2,j}$ ,  $k_j = h_{4,j}h_{2,j+2}$ ,  $l_j = h_{4,j}h_{4,j+1}h_{2,j}$  and

$$l'_j = h_{4,j}h_{4,j+1}h_{2,j+1} + h_{4,j+1}h_{4,j+2}h_{2,j}$$

$$A = h_{4,0}h_{4,1}h_{4,2}h_{2,0}h_{2,1}h_{2,2} = -g_0g_1g_2.$$

*Proof.* From (5.4), it is easy to see that  $d_1(h_{4,j}h_{2,j}) = 0$ ,  $d_1(h_{4,j}h_{2,j+2}) = 0$  and from  $d_1(h_{4,j+1}) = h_{2,j+1}h_{2,j+3} = h_{2,j+1}h_{2,j}$  we see that  $d_1(h_{4,j}h_{4,j+1}h_{2,j}) = 0$ . These gives the cohomology classes  $g_j$ ,  $k_j$  and  $l_j$ . From

$$d_1(h_{4,j}h_{4,j+1}h_{2,j+1}) = -h_{2,j}h_{2,j+2}h_{4,j+1}h_{2,j+1} = -h_{4,j+1}h_{2,j}h_{2,j+1}h_{2,j+2}$$

$$d_1(h_{4,j+1}h_{4,j+2}h_{2,j}) = h_{4,j+1}h_{2,j+2}h_{2,j+4}h_{2,j} = h_{4,j+1}h_{2,j}h_{2,j+1}h_{2,j+2}$$

we get  $l'_j$ . A routine computation shows that  $H^*(E[h_{2,j}, h_{4,j}|j \in Z/3]) = \mathcal{C}$ .  $\square$

**Theorem 5.6** *The May  $E_2$ -term for  $H^{*,*}S(3, 2)$  at  $p = 3$  is the  $E_\infty$ -term, thus  $H^{*,*}S(3, 2)$  is the tensor product of  $E[h_{3,j}|j \in Z/3]$  and  $\mathcal{C}$ .*

*Proof.* It is easy to see that  $h_{2,j}$  and  $h_{3,j}$  are infinite cycles. To prove that all the higher May differentials are trivial, consider the May filtration of each generator in the  $E_2$ -term and the differentials

$$d_r : E_r^{s,t,M}S(3, 2) \longrightarrow E_r^{s+1,t,M-r}S(3, 2).$$

One has

$$\begin{aligned} g_j &\in E_2^{2,*,10}S(3, 2) & k_j &\in E_2^{2,*,10}S(3, 2) \\ l_j &\in E_2^{3,*,17}S(3, 2) & l'_j &\in E_2^{3,*,17}S(3, 2) \\ h_{2,j} &\in E_2^{1,*,3}S(3, 2) & h_{3,j} &\in E_2^{1,0,5}S(3, 2). \end{aligned}$$

The May filtration of  $g_j$  and  $k_j$  are 10. Beside, it is easy to check that each generator in the 3rd dimension  $E_2^{3,*,*}S(3, 2)$  listed as below

$$l_j, \quad l'_j, \quad k_j h_{2,j}, \quad g_j h_{3,i}, \quad k_j h_{3,i}, \quad h_{3,i} h_{3,j} h_{2,k}, \quad h_{3,0} h_{3,1} h_{3,2}$$

has May filtration  $\geq 10$ . Thus  $g_j$  and  $k_j$  are infinite cycles. Similarly one can prove that  $l_j$  and  $l'_j$  are infinite cycles. This complete the proof.  $\square$

**5.3. The cohomology of  $S(4, 2)$  at the primes  $p > 3$ .** In this case,  $s_0 = 5$  and the reduced May's  $E_1$ -term is

$$\tilde{E}_1^{*,*,*} S(4, 2) = E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j} | j \in Z/4].$$

To compute the  $E_2$ -term, we set a filtration on the exterior algebra  $E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j} | j \in Z/4]$  as follows:

$$F^k = \bigoplus_{0 \leq r \leq k} Z/p\{h_{5,j_1} \cdots h_{5,j_r}\} \otimes E[h_{2,j}, h_{3,j}, h_{4,j} | j \in Z/4]$$

where  $h_{5,j_1} \cdots h_{5,j_r}$ 's are the generators of the  $r$ -dimensional module of the exterior algebra  $E[h_{5,j} | j \in Z/4]$ . This filtration gives rise to a spectral sequence with

$$E_0^k = F^k / F^{k-1} = Z/p\{h_{5,j_1} \cdots h_{5,j_k}\} \otimes E[h_{2,j}, h_{3,j}, h_{4,j} | j \in Z/4].$$

The  $E_1$ -term of this spectral sequence is

$$E_1^k = Z/p\{h_{5,j_1} \cdots h_{5,j_r}\} \otimes H^* E[h_{2,j}, h_{3,j}, h_{4,j} | j \in Z/4],$$

and the differentials are given by

$$d_r : E_r^k \longrightarrow E_r^{k-r}.$$

By a routine computation, we get

**Theorem 5.7** *The cohomology of  $E[h_{2,j}, h_{3,j}, h_{4,j} | j \in Z/4]$  is the tensor product of  $E[h_{3,j}, \rho_0, \rho_1]$  and  $\aleph$ , where*

$$\rho_0 = h_{4,0} + h_{4,2}, \quad \rho_1 = h_{4,1} + h_{4,3},$$

and  $\aleph$  is the direct sum of the modules generated by the following cohomology classes:

$$\begin{array}{llll} 1; & h_{2,j}; & e_j = h_{2,j}h_{2,j+1}, & g_j = h_{4,j}h_{2,j}; \\ h_{2,j}g_{j+1}, & h_{2,j}g_{j+2}, & h_{2,j}g_{j+3}; & \\ g_jg_{j+1}, & e_jg_{j+2}; & h_{2,j}g_{j+1}g_{j+2}; & e_0g_2g_3 \end{array}$$

with  $j \in Z/4$ . Beside, we also have the following relations:

$$h_{2,i}h_{2,i+2} = 0, \quad h_{2,i}g_{i+2} = h_{2,i+2}g_i, \quad h_{2,i}g_{i+2}g_{i+3} = h_{2,i+2}g_{i+3}g_i.$$

With the add of a personal computer, we compute that

**Theorem 5.8** *The cohomology of the exterior algebra  $E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j}]$  has Poincaré series*

$$(1+t)^4(1+6t+18t^2+59t^3+92t^4+176t^5+161t^6+176t^7+92t^8+59t^9+18t^{10}+6t^{11}+t^{12}).$$

The ranks at each cohomological dimension are listed as

$$\begin{array}{cccccccccccccccc} 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \cdots & 16, \\ 1, & 10, & 48, & 171, & 461, & 976, & 1671, & 2303, & 2558, & 2303, & \cdots & 1 \end{array}$$

From the collapse Theorem 4.9, we claim that the MSS for the cohomology of  $S(4, 2)$  collapse at  $E_2$ -term.

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